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Change-point detection in panel data

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We consider N panels and each panel is based on T observations. We are interested to test if the means of the panels remain the same during the observation period against the alternative that the means change at an unknown time. We provide tests which are derived from a likelihood argument and they are based on the adaptation of the CUSUM method to panel data. Asymptotic distributions are derived under the no change null hypothesis and the consistency of the tests are proven under the alternative. The asymptotic results are shown to work in case of small and moderate sample sizes via Monte Carlo simulations.

Keywords: Panel date; change in the mean; linear processes; weak convergence; CUSUM process.
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1. INTRODUCTION AND RESULTS

One of the tools to analyse large, high-dimensional data sets is the panel data model. The focus of this article is to test for possible changes in the location (mean) parameter of panel data. We assume that we study N panels and we have T observations in each panel. We define our model as

$$X_{i,t} = \mu_i + \delta_i I\{t > t_0\} + e_{i,t}, \quad 1 \le i \le N, 1 \le t \le T, \tag{1}$$

where $Ee_{i,t} = 0$ for all i and t. According to (1), μ_i changes to $\mu_i + \delta_i$ in case of panel i at time t_0. The parameter t_0, the time of change, is unknown. Both T and N are assumed to be large. In this article, we wish to test that the location parameter μ_i will not change during the observation period, i.e.

$$H_0 : \delta_i = 0 \quad \text{for all } 1 \le i \le N.$$

There is an ever increasing literature to test the structural stability of univariate as well as multi-variate models (cf. Csörgő and Horváth, 1997; Brodsky and Darkhovskii, 2000) but much less is known on the stability of panel models. Change point detection in panel data can be viewed as a structural stability problem in high dimensional time series. Joseph and Wolfson (1992, 1993) initiated change point models for panel data. Im *et al.* (2005) and Bai and Carrion-i-Silvestre (2009) discuss the analysis of panel data with possible change points in case of stationary and non-stationary (random walk) errors. Atak *et al.* (2011) use a panel data model to detect changes in the climate in the United Kingdom. Bai (2010) uses the least squares and the quasi-maximum likelihood method to estimate the time of change (t_0) assuming *a priori* that a change has occurred, i.e. H_0 does not hold. We follow Bai's (2010) model in our study.

Using a quasi-maximum likelihood argument, Bai (2010) estimated t_0, the location of the time of change, by the location of the maximum of the absolute value of

$$\bar{V}_{N,T}(x) = \frac{1}{N^{1/2}} \sum_{i=1}^{N} \left\{ \frac{1}{\sigma_i^2} Z_{T,i}^2(x) - \frac{\lfloor Tx \rfloor (T - \lfloor Tx \rfloor)}{T^2} \right\}, \quad 0 \le x \le 1, \tag{2}$$

where $\lfloor \cdot \rfloor$ denotes the integer part,

$$Z_{T,i}(x) = \frac{1}{T^{1/2}} \left(S_{T,i}(x) - \frac{\lfloor Tx \rfloor}{T} S_{T,i}(1) \right), \quad 0 \le x \le 1 \tag{3}$$

with

$$S_{T,i}(x) = \sum_{t=1}^{\lfloor Tx \rfloor} X_{i,t}, \quad 0 \le x \le 1,$$

and the σ_i^2's are some suitably chosen standardization constants [cf. (10)]. Using our notation, the estimator used by Bai (2010) can be

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written as

$$\hat{t}_0 = \arg\max_{1 \le t \le T-1} \left\{ \sum_{i=1}^{N} \frac{Z_{T,i}^2(t/T)}{(t(T-t))} \right\}.$$

In this article, we use a model where the innovations (errors) form a linear process:

$$e_{i,t} = \sum_{\ell=0}^{\infty} c_{i,\ell}\varepsilon_{i,t-\ell}, \quad 1 \le i \le N, 1 \le t \le T.$$

For the moment, we assume the following regularity conditions:

$$\text{the sequences } \{\varepsilon_{i,t}, -\infty < t < \infty\} \text{ are independent of each other} \tag{4}$$

$$\text{for every } i \text{ the variables } \{\varepsilon_{i,t}, -\infty < t < \infty\} \text{ are i.i.d. .} \tag{5}$$

Assumption (4) means that the panels are independent. The case of dependent panels will be briefly discussed in Section 3. It is easy to see that under our conditions, the process $\bar{V}_{N,T}$ does not depend on $\text{var}\varepsilon_{i,0}$ so it can be assumed that the variance of the innovations is 1:

$$E\varepsilon_{i,0} = 0, \quad E\varepsilon_{i,0}^2 = 1 \quad \text{and} \quad E|\varepsilon_{i,0}|^{\kappa} < \infty. \tag{6}$$

The distributions of the $\varepsilon_{0,j}$'s can be very different, but the next condition requires that the average of the high moments is bounded:

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} E|\varepsilon_{i,0}|^{\kappa} < \infty. \tag{7}$$

The choice of κ will be specified in Theorems 1–3 as $\kappa > 4$ and $\kappa = 8$ in Theorem 4. Of course, (7) holds, if the moments $E|\varepsilon_{i,0}|^{\kappa}$ are uniformly bounded. However, condition (7) allows that some of the $\varepsilon_{i,0}$'s and hence the error terms possess κth order moments of large magnitude as long as (7) still holds.

The errors in each panel are stationary linear sequences and their distributions depend on the panel. The coefficients in the definition of the linear sequences have the following properties:

$$|c_{i,\ell}| \le c_0(\ell+1)^{-\alpha} \quad \text{for all } 1 \le i \le N, 0 \le \ell < \infty \quad \text{with some } c_0 \text{ and } \alpha > 2 \tag{8}$$

and

$$\text{there is } \delta > 0 \text{ such that } a_i^2 \ge \delta^2 \text{ with } a_i = \sum_{\ell=0}^{\infty} c_{i,\ell} \text{ for all } 1 \le i \le N. \tag{9}$$

Since

$$\lim_{T\to\infty} \frac{1}{T} E\left(\sum_{t=1}^{T} e_{i,t}\right)^2 = \sigma_i^2, \quad 1 \le i \le N, \tag{10}$$

we obtain immediately from assumptions (6) and (9) that $a_i^2 = \sigma_i^2$ and

$$\sigma_i^2 \ge \delta^2 \quad \text{for all } 1 \le i \le N, \tag{11}$$

i.e. we have a common lower bound for the long-run variances of each panel. The next condition is on the connection between the number of panels (N) and the length of the observed time series in each panel (T):

$$\frac{N}{T^2} \to 0. \tag{12}$$

Condition (12) allows that the number of panels is larger than the number of the observations in the individual panels.

Let $\xrightarrow{\mathcal{D}[0,1]}$ denote the weak convergence of stochastic processes in the Skorokhod space $\mathcal{D}[0,1]$.

Theorem 1. *If* H_0, (4), (5), (8), (9), (12) *hold and* (6), (7) *are satisfied with some* $\kappa > 4$, *then*

$$\bar{V}_{N,T}(x) \xrightarrow{\mathcal{D}[0,1]} \Gamma(x),$$

where $\Gamma(x)$ is a Gaussian process with $E\Gamma(x) = 0$ and $E\Gamma(x)\Gamma(y) = 2x^2(1-y)^2$, if $0 \le x \le y \le 1$.

Remark 1. Checking the covariance functions, one can easily verify that

$$\{\Gamma(x), 0 \le x \le 1\} \stackrel{\mathcal{D}}{=} \{\sqrt{2}(1-x)^2 W\left(\frac{x^2}{(1-x)^2}\right), 0 \le x \le 1\}, \tag{13}$$

where $\{W(y), y \ge 0\}$ is a Wiener process (standard Brownian motion). It is well known that the CUSUM process (standardized by the long-run variance) converges weakly to a Brownian bridge assuming weak dependence. Under the conditions of Theorem 1 for each i, the process $Z_{T,i}(x)$ converges to a Brownian bridge. So it is interesting to compare (13) with $(1-x)W(x/(1-x))$ which defines a Brownian bridge.

The next result illustrates that condition (12) is optimal. To state the result, we need further notation. We introduce the linear process

$$e^*_{i,t} = \sum_{\ell=1}^{\infty} c^*_{i,\ell}\varepsilon_{i,t-\ell} \quad \text{with} \quad c^*_{i,\ell} = \sum_{k=\ell+1}^{\infty} c_{i,k}. \tag{14}$$

The process $e^*_{i,t}$ appears in the Phillips and Solo (1992) representation of the sums of the $e_{i,t}$'s [cf. (31)]. Let

$$g(x) = 2[1 - x + x^2]\left\{\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\text{var}(e^*_{i,0}) - \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}a_i c^*_{i,0}\right\}.$$

Theorem 2. *If H_0, (4), (5), (8) and (9) hold and (6), (7) are satisfied with some $\kappa > 4$, then*

$$\bar{V}_{N,T}(x) - \left(\frac{N^{1/2}}{T}\right)g(x) \stackrel{\mathcal{D}[0,1]}{\longrightarrow} \Gamma(x),$$

as $\min(N,T) \to \infty$, where $\Gamma(x)$ is defined in Theorem 1.

Theorem 2 means, if $N/T^2 \to c \neq 0$, there is a non-disappearing drift term in the limit.

Remark 2. We wish to point out that there are no assumptions on the μ_i's, and they can be random as well as non-random terms.

Next, we provide a result which can be easily used to show the asymptotic consistency of tests based on Theorem 1.

Theorem 3. *If (4), (5), (8), (9), (12) hold and (6), (7) are satisfied with some $\kappa > 4$ and $t_0 = t_0(T)$,*

$$0 < \liminf_{T\to\infty}\frac{t_0}{T} \le \limsup_{T\to\infty}\frac{t_0}{T} < 1,$$

$$\frac{T}{N^{1/2}}\sum_{i=1}^{N}\delta_i^2 \longrightarrow \infty, \tag{15}$$

as $\min(N,T) \to \infty$, then

$$\sup_{0\le x\le 1}|\bar{V}_{N,T}(x)| \stackrel{P}{\longrightarrow} \infty.$$

Theorem 1 yields that $\sup_{0\le x\le 1}|\bar{V}_{N,T}(x)|$ and $\int_0^1 \bar{V}^2_{N,T}(x)\mathrm{d}x$ converge in distribution to $\sup_{0\le x\le 1}|\Gamma(x)|$ and $\int_0^1 \Gamma^2(x)\mathrm{d}x$, respectively. So large values of $\sup_{0\le x\le 1}|\bar{V}_{N,T}(x)|$ or $\int_0^1 \bar{V}^2_{N,T}(x)\mathrm{d}x$ indicate that the null hypothesis is violated. The asymptotic critical values can be computed from Theorem 1. Theorem 3 gives conditions for the consistency of both tests. Condition (15) covers quite a large spectrum of alternatives. The test is sensitive to fixed changes in relatively few panels, and at the same time, it is sensitive to relatively small changes in a large number of panels. This means that the tests will detect instability of the model in case of small changes in several panels as well as relatively large changes in few panels. We also would like to point out that Theorem 3 can be easily extended to the multiple changes model.

2. ESTIMATION OF LONG-RUN VARIANCES

Since the parameters σ_i in the defintion of $\bar{V}_{N,T}$ are unknown, we need to replace them with some suitable estimators. Hence, we define

$$V_{N,T}(x) = \frac{1}{N^{1/2}} \sum_{i=1}^{N} \left\{ \frac{1}{\hat{\sigma}_T^2(i)} Z_{T,i}^2(x) - \frac{\lfloor Tx \rfloor (T - \lfloor Tx \rfloor)}{T^2} \right\}, \quad 0 \le x \le 1, \tag{16}$$

where the long-run variance of $T^{-1/2} S_{T,i}(1)$ is estimated by $\hat{\sigma}_T^2(i)$. If for any i, the errors $\{e_{i,t}, 1 \le t \le T\}$ are i.i.d., we can use the sample variance

$$\hat{\sigma}_T^2(i) = \frac{1}{T-1} \sum_{t=1}^{T} (X_{i,t} - \bar{X}_T(i))^2, \quad \bar{X}_T(i) = \frac{1}{T} \sum_{t=1}^{T} X_{i,t} \tag{17}$$

to estimate the variance of $T^{-1/2} S_{T,i}(1)$ in the ith panel. If independence cannot be assumed, a kernel estimator is used:

$$\hat{\sigma}_T^2(i) = \frac{1}{T} \sum_{t=1}^{T} (X_{i,t} - \bar{X}_T(i))^2 + 2 \sum_{s=1}^{T-1} K\left(\frac{s}{h}\right) \hat{\gamma}_{T,s}(i), \tag{18}$$

where

$$\hat{\gamma}_{T,s}(i) = \frac{1}{T-s} \sum_{t=1}^{T-s} (X_{i,t} - \bar{X}_T(i))(X_{i,t+s} - \bar{X}_T(i))$$

is the sample correlation of lag s in the ith panel. The function K is the kernel in the definition of $\hat{\sigma}_T^2(i)$ in (18) and $h = h(T)$ is the window. For a discussion on kernel estimators, we refer to Taniguchi and Kakizawa (2000) and Brockwell and Davis (2006). Throughout this article, we assume the following conditions on the kernel estimator:

$$K(0) = 1 \tag{19}$$

$$K(u) = 0 \text{ if } |u| > a \text{ and } K(u) \text{ is Lipschitz continous on } [-a, a] \text{ with some a} > 0 \tag{20}$$

$$\begin{aligned} &K \text{ has } \nu \text{ bounded derivatives in a neighbourhood of 0 and the first} \\ &\nu - 1 \text{ derivatives of } K \text{ are 0 at 0, where } \nu \ge 1 \text{ is an integer} \end{aligned} \tag{21}$$

and

$$h = h(T) \to \infty \text{ and } \frac{h}{T} \to 0 \text{ as } T \to \infty. \tag{22}$$

We note that the 'flat top' kernel satisfies (21) for all $\nu \ge 1$. Assumption (21) is needed to have a very small bias of the estimators $\hat{\sigma}_T^2(i)$. We see in Section 3 that even very small changes to the model in (1) alter the asymptotics for $\bar{V}_{N,T}$. Similarly, the estimator $\hat{\sigma}_T^2(i)$ must be very close to σ_i^2 to claim that $V_{N,T}$ and $\bar{V}_{N,T}$ have the same asymptotic distribution.

The next condition is on the connection between the number of panels (N), the length of the observed time series in each panel (T) and the bandwidth (h):

$$\frac{Nh^2}{T^2} \to 0 \quad \text{and} \quad \frac{N^{1/2}}{h^\tau} \to 0, \quad \text{where } \tau = \min(\nu, \alpha - 1). \tag{23}$$

As before, assumption (23) allows to have short time-series in a much larger number of panels.

Theorem 4. *If H_0, (4), (5), (8), (9), (19)–(23) hold and (6), (7) are satisfied with $\kappa = 8$, then*

$$V_{N,T}(x) \xrightarrow{\mathcal{D}[0,1]} \Gamma(x), \tag{24}$$

where $\Gamma(x)$ is defined in Theorem 1.

3. DEPENDENT PANELS

In some applications, it cannot be assumed that the panels are independent, and therefore we provide a simple modification of the model in (1). We introduce the dependence between the panels through the term ζ_t:

$$X_{i,t} = \mu_i + \delta_i I\{t \ge t_0\} + \phi_i \zeta_t + e_{i,t}, \quad 1 \le i \le N, 1 \le t \le T, \tag{25}$$

where we assume that

$$\{\zeta_t, 1 \le t < \infty\} \quad \text{and} \quad \{e_{i,t}, 1 \le t < \infty\} \text{ are independent.} \tag{26}$$

Instead of specifying the structure of the ζ_t's, we require that they satisfy the functional central limit theorem:

$$\frac{1}{T^{1/2}} \sum_{t=1}^{\lfloor Tx \rfloor} \zeta_t \xrightarrow{\mathcal{D}[0,1]} W(x), \tag{27}$$

where $W(x)$ is a Wiener process (standard Brownian motion). The scaling constants ϕ_i explain how much influence ζ_t has on the ith panel. Interestingly, even small dependence between the panels changes the asymptotics for $\bar{V}_{N,T}$ and $V_{N,T}$ under H_0. For example, if

$$\phi_i = \phi_{i,N} = \frac{\xi_i}{N^{\rho_i}} \quad \text{with some } \xi_i \ne 0 \quad \text{and } 0 \le \rho_i < 1/4,$$

and $\{\xi_i\}$ is a bounded sequence, then we have

$$\sup_{0 \le x \le 1} |\bar{V}_{N,T}(x)| \xrightarrow{P} \infty. \tag{28}$$

The proof of (28) in Section 8 shows that

$$\frac{N^{1/2}}{\sum_{i=1}^{N} \xi_i^2} \bar{V}_{N,T}(x) \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where $B(x)$ is a Brownian bridge. If $\rho_i = 1/4$ in (28), then

$$\bar{V}_{N,T}(x) \xrightarrow{\mathcal{D}[0,1]} \Gamma(x) + \xi_0 B^2(x) \quad \text{with } \xi_0 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\xi_i^2}{\sigma_i^2}, \tag{29}$$

and

$$\frac{1}{N^{1/2}} \sum_{i=1}^{N} \left\{ \frac{1}{\sigma_i^2 + \phi_i^2} Z_{T,i}^2(x) - \frac{\lfloor Tx \rfloor (T - \lfloor Tx \rfloor)}{T^2} \right\} \xrightarrow{\mathcal{D}[0,1]} \Gamma(x) + \xi_0^2 (B^2(x) - x(1-x)), \tag{30}$$

where $B(x)$ stands for a Brownian bridge.

If $\rho_i > 1/4$ for all i, then Theorem 4 still holds. Roughly speaking, the dependence has no effect on the test statistic if the correlation between the panels is less than $N^{-1/2}$.

4. SIMULATIONS

We used Monte Carlo simulations to check if Theorems 1 and 4 provide good approximations in case of small and moderate panels and sample sizes. Our test statistic is $\sup_x |V_{N,T}(x)|$, where $V_{N,T}$ is defined in (16). [Similar results were obtained for $\sup_x |\bar{V}_{N,T}(x)|$, where $\bar{V}_{N,T}$ is given in (2).] First, simulating Brownian motions we obtained z_α from the equation

$$P\{ \sup_{0 \le x \le 1} |\Gamma(x)| > z_\alpha \} = \alpha,$$

and the values for $\alpha = 0.1$,0.05 and 0.01 are given in Table 1. Next, we studied if z_α is a good approximation for $z_\alpha(N,T)$, where $z_\alpha(N,T)$ is defined by

$$z_\alpha(N,T) = \inf\left(y : P\{ \sup_{0 \le x \le 1} |V_{N,T}(x)| > y \} \le \alpha \right).$$

The results are given in Table 2 when the panels are independent of each other, and the panels are based on independent identically distributed random variables. We considered standard normal, χ^2 and t distributions with 5 degrees of freedoms. The variances were estimated by the corresponding sample variances of the panels.

Next, we studied the effect of independence on the critical values. The long-run variance was estimated by $\hat{\sigma}_T^2(i)$ of (18) with the flat top kernel

Table 1. Critical values of $\sup_{0 \le x \le 1} |\Gamma(x)|$

0.1	0.05	0.01
0.796	0.894	1.145

Table 2. Simulated critical values of $\sup_{0<x<1}|V_{N,T}(x)|$ based standard normal, χ^2_5 and t_5 errors

	Normal			χ^2_5			t_5		
N/T	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01
50/50	0.81	0.91	1.15	0.80	0.90	1.09	0.78	0.87	1.17
100/50	0.80	0.89	1.11	0.80	0.89	1.14	0.78	0.88	1.06
100/100	0.83	0.94	1.20	0.82	0.92	1.13	0.84	0.93	1.10
200/100	0.84	0.930	1.13	0.83	0.92	1.11	0.82	0.90	1.12

$$K(u) = \begin{cases} 1, & \text{if } |u| \le 1/2 \\ 2(1-|u|), & \text{if } 1/2 \le |u| \le 1 \\ 0, & \text{if } |u| \ge 1. \end{cases}$$

We tried several values for h and $h \in [2.5,5]$ worked well. The results are reported in Table 3 for AR(1) process with standard normal innovations and $h = 3$. Table 4 contains the results when the innovations are independent t variables with 5 degrees of freedom.

Table 2 is based on the assumption that the panels are formed from independent observations so the sample variance can be used to estimate the variance of the panels. Table 3 does not assume that the observations are independent so the long-run variance estimator of (18) is used. Comparing Table 2 and Table 3 with $\rho = 0$ illustrates that the estimation of the long-run variance reduces the accuracy of the limit results.

We considered the power of the test very briefly. The size of the changes was independent uniform on $[-1,1]$ or $[-1/2,1/2]$ in 50% as well as in 10% of the panels. The time of change is $t_0 = T/2$ The observed frequency of rejections are reported at the 5% siginificance levels in case of independent standard normal errors (Table 5) and AR(1) process with $\rho = 0.1$ and $\rho = 0.2$ with independent standard normal innovations (Tables 6 and 7). Clearly, the test has very good power even in case of small changes in few panels. All simulations are based on 2000 replications.

To illustrate the applicability of our results, we used data from the World Income Inequality Database at `http://wider.unu.edu/research/Database/en_GB/wiid/`. The original data set contains the Gini coefficients in percentage points and source information for 159 countries. (The Gini index is used to measure the inequality of wealth.) Unfortunately, a large amount of data is missing especially before the 1980s. We selected 33 countries from 1987 to 2006, including European countries (United Kingdom, Spain and so on), Australia, United States, South American countries (Argentina, Venezuela and so on), China and Taiwan. Each country has at least 16 recorded observations. Missing observations were replaced using linear interpolation if more than one point was given for one year, they were replaced by the average. So our analysis is based on $N = 33$ and $T = 20$. Since T is small, the long-run variance estimator in (18) might not be accurate. Hence, we used the sample variance in (17) and the long-run variance estimator

Table 3. Simulated critical values of $\sup_{0<x<1}|V_{N,T}(x)|$ based AR(1) processes standard normal innovations

	$\rho = 0$			$\rho = 0.1$			$\rho = 0.3$			$\rho = 0.5$		
N/T	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01
50/50	0.97	1.11	1.42	0.89	1.03	1.26	0.93	1.05	1.25	1.23	1.36	1.64
100/50	1.01	1.14	1.57	0.93	1.05	1.31	0.99	1.11	1.33	1.40	1.54	1.80
100/100	0.89	1.00	1.23	0.88	0.98	1.20	0.98	1.11	1.32	1.49	1.68	1.95
200/100	0.92	1.04	1.25	0.89	0.99	1.23	1.09	1.20	1.43	1.76	1.88	2.18

Table 4. Simulated critical values of $\sup_{0<x<1}|V_{N,T}(x)|$ based AR(1) processes with t_5 innovations

	$\rho = 0$			$\rho = 0.1$			$\rho = 0.3$			$\rho = 0.5$		
N/T	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01
50/50	0.96	1.09	1.43	0.86	0.97	1.29	0.92	1.05	1.26	1.22	1.33	1.61
100/50	0.99	1.13	1.40	0.92	1.03	1.29	0.96	1.07	1.36	1.39	1.52	1.80
100/100	0.88	1.00	1.21	0.88	0.98	1.21	0.99	1.10	1.36	1.50	1.64	1.98
200/100	0.90	1.02	1.24	0.89	0.99	1.22	1.07	1.18	1.40	1.76	1.92	2.20

Table 5. Empirical rejection percentages for $\sup_{0<x<1}|V_{N,T}(x)|$ at 5% significance level with independent standard normal errors

	$U[-1/2,1/2]$		$U[-1,1]$	
N/T	50%	10%	50%	10%
50/50	60.3	11.9	99.9	41.5
100/50	80.3	14.6	100	57.6
100/100	99.9	31.1	100	92.6
200/100	100	47.8	100	99.9

Table 6. Empirical rejection percentages for $\sup_{0<x<1}|V_{N,T}(x)|$ at 5% significance level in case of AR(1) processes with $\rho = 0.1$ and standard normal innovations

	$U[-1/2,1/2]$		$U[-1,1]$	
N/T	50%	10%	50%	10%
50/50	52.2	14.2	99.3	32.5
100/50	72.3	23.0	100	49.3
100/100	98.5	27.0	100	83.2
200/100	100	44.4	100	97.6

Table 7. Empirical rejection percentages for $\sup_{0<x<1}|V_{N,T}(x)|$ at 5% significance level in case of AR(1) processes with $\rho = 0.2$ and standard normal innovations

	$U[-1/2,1/2]$		$U[-1,1]$	
N/T	50%	10%	50%	10%
50/50	42.0	12.8	97.4	27.2
100/50	62.1	21.1	100	42.8
100/100	95.2	24.9	100	77.7
200/100	99.9	42.4	100	95.0

in (18) with several choices of h. In all cases, the no change in the mean null hypothesis was rejected at least 1% significance level. Visual inspection of the panels shows an increase in the Gini index in nearly all countries used in our example except countries in Northern Europe (Figure 1). Figure 2 is more typical where jumps in the Gini indices can be observed. Our estimator for the time of change is the location of the maximum of the test statistics. We obtain 1992 as the estimator for the time of change which can be visually observed on Figure 2.

5. PROOF OF THEOREM 1

Lemma 1. *If* (4)–(9) *and* (12) *hold, then the finite dimensional distributions of* $\bar{V}_{N,T}(x)$ *converge to that of* $\Gamma(x)$*, where* $\Gamma(x)$ *is defined in Theorem* 1.

Proof. According to Phillips and Solo (1992) (cf. Bai, 1994, p. 470) we have that

$$\sum_{t=1}^{k} e_{i,t} - \frac{k}{T}\sum_{t=1}^{T} e_{i,t} = a_i\left(\sum_{t=1}^{k}\varepsilon_{i,t} - \frac{k}{T}\sum_{t=1}^{T}\varepsilon_{i,t}\right) + \eta_{i,k}, \tag{31}$$

where a_i is defined in (9),

$$\eta_{i,k} = \eta_{i,k}(T) = \left(1 - \frac{k}{T}\right)e^{*}_{i,0} - e^{*}_{i,k} + \frac{k}{T}e^{*}_{i,T},$$

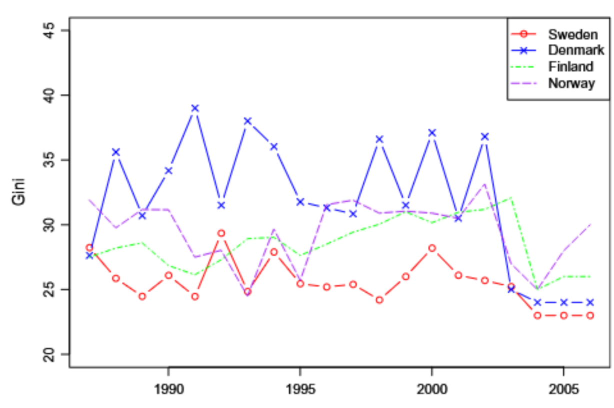

Figure 1. The Gini indices (in percentages) for four Northern European countries

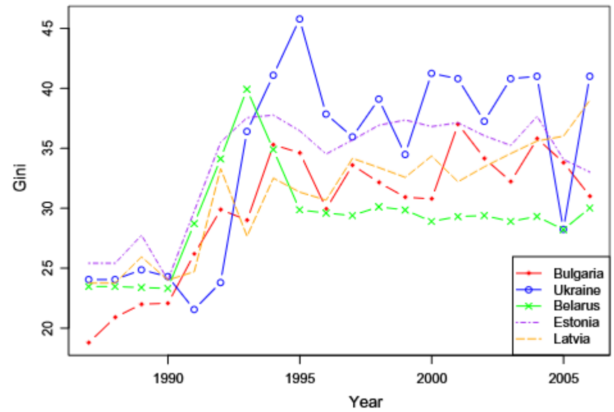

Figure 2. The Gini indices (in percentages) for five former socialist countries

where the $e^*_{i,t}$'s are defined in (14). We recall that according to (6), $a_i^2 = \sigma_i^2$. Let

$$Q_{T,i}(x) = \frac{1}{T^{1/2}}\left(\sum_{t=1}^{\lfloor Tx\rfloor}\varepsilon_{i,t} - \frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}\varepsilon_{i,t}\right).$$

Now, (31) yields

$$Z^2_{T,i}\left(\frac{k}{T}\right) = a_i^2 Q^2_{T,i}\left(\frac{k}{T}\right) + 2a_i Q_{T,i}\left(\frac{k}{T}\right)T^{-1/2}\eta_{i,k} + \frac{\eta^2_{i,k}}{T}. \tag{32}$$

Since $Q_{T,i}$ is a tied-down sum of independent identically distributed random variables, we obtain immediately that

$$a_i^2 E Q^2_{T,i}\left(\frac{k}{T}\right) = \sigma_i^2\left(1-\frac{k}{T}\right)\frac{k}{T}. \tag{33}$$

Using the definition of $e^*_{i,j}$ we conclude that

$$EQ_{T,i}\left(\frac{k}{T}\right)e^*_{i,0} = 0, \quad EQ_{T,i}\left(\frac{k}{T}\right)\frac{1}{T^{1/2}}e^*_{i,k} = \frac{1}{T}\left(1-\frac{k}{T}\right)c^*_{i,0}E\varepsilon^2_{i,0},$$
$$EQ_{T,i}\left(\frac{k}{T}\right)\frac{k}{T^{3/2}}e^*_{i,T} = \frac{k}{T^2}\left(\sum_{\ell=T-k}^{T}c^*_{i,\ell} - \frac{k}{T}c^*_{i,0}\right)E\varepsilon^2_{i,0}$$

and by the stationarity of $e^*_{i,j}$ we have

$$E\frac{1}{T}\eta^2_{i,k} \le \frac{8}{T}\left(E(e^*_{i,0})^2 + E(e^*_{i,k})^2 + E(e^*_{i,T})^2\right) \le \frac{24}{T}E(e^*_{i,0})^2.$$

It follows from (8) that

$$\limsup_{i\to\infty}\sum_{\ell=0}^{\infty}|c^*_{i,\ell}| < \infty$$

and therefore

$$E(e^*_{i,0})^2 \le C_1 E\varepsilon^2_{i,0}$$

with some constant C_1. Thus, we obtain by (11)

$$\sup_{0\le x\le 1}\frac{1}{N^{1/2}}\left|\sum_{i=1}^{N}\left\{\frac{1}{\sigma_i^2}EZ^2_{T,i}(x) - \frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right\}\right| \le C_2\frac{N^{1/2}}{T}\frac{1}{N}\sum_{i=1}^{N}E\varepsilon^2_{i,0}$$

with some constant C_2 and therefore on account of (6) and (12)

$$\sup_{0\le x\le 1}\frac{1}{N^{1/2}}\left|\sum_{i=1}^{N}\left\{\frac{1}{\sigma_i^2}EZ^2_{T,i}(x) - \frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right\}\right| = o(1). \tag{34}$$

Next, we show that for all k

$$\bar{V}_{N,T}\left(\frac{k}{T}\right) - R_{N,T}\left(\frac{k}{T}\right) = o_P(1), \tag{35}$$

where

$$R_{N,T}(x) = \frac{1}{N^{1/2}}\sum_{i=1}^{N}\left\{Q_{T,i}^2(x) - \frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right\}. \tag{36}$$

In light of (34) it is enough to prove that

$$\max_{1\le k\le T}\frac{1}{N}\sum_{i=1}^{N}E\left(\frac{1}{\sigma_i^2}Z_{T,i}^2\left(\frac{k}{T}\right) - Q_{T,i}^2\left(\frac{k}{T}\right)\right)^2 = o(1). \tag{37}$$

Using again (32) we obtain that

$$\begin{aligned}E(Z_{T,i}^2\left(\frac{k}{T}\right) - a_i^2 Q_{T,i}^2\left(\frac{k}{T}\right))^2 &= E(2a_i Q_{T,i}\left(\frac{k}{T}\right)T^{-1/2}\eta_{i,k} + \eta_{i,k}^2/T)^2\\ &\le 16E\left\{a_i^2 Q_{T,i}^2\left(\frac{k}{T}\right)\eta_{i,k}^2/T + \eta_{i,k}^4/T^2\right\}\\ &\le C_3\left\{\frac{1}{T^2}E\varepsilon_{i,0}^4 + \frac{1}{T}(E\varepsilon_{i,0}^4)^{1/2}(EQ_{T,i}^4\left(\frac{k}{T}\right))^{1/2}\right\}\end{aligned}$$

with some constant C_3. By the Rosenthal inequality (cf. Petrov, 1995, p. 59), we conclude

$$EQ_{T,i}^4\left(\frac{k}{T}\right) \le \frac{C_4}{T^2}\left\{TE\varepsilon_{i,0}^4 + T^2(E\varepsilon_{i,0}^2)^2\right\},$$

where C_4 is a constant and therefore by the Cauchy–Schwarz inequality we have

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}E\left(\frac{1}{\sigma_i^2}Z_{T,i}^2\left(\frac{k}{T}\right) - Q_{T,i}^2\left(\frac{k}{T}\right)\right)^2 &\le C_5\left\{\frac{1}{T^{3/2}}\frac{1}{N}\sum_{i=1}^{N}E\varepsilon_{i,0}^4 + \frac{1}{T}\frac{1}{N}\sum_{i=1}^{N}(E\varepsilon_{i,0}^4)^{1/2}E\varepsilon_{i,0}^2\right\}\\ &\le C_6\frac{1}{T}\frac{1}{N}\sum_{i=1}^{N}E\varepsilon_{i,0}^4,\end{aligned}$$

completing the proof of (37).

Let $0 < x_1 < x_2 < \cdots < x_K < 1$ and $\lambda_1, \lambda_2, \ldots, \lambda_K$ be constants and introduce

$$L_{T,i} = \sum_{\ell=1}^{K}\lambda_\ell(Q_{T,i}^2(x_\ell) - EQ_{T,i}^2(x_\ell)).$$

Since $Q_{T,i}$ is a tied-down sum of independent, identically distributed random variables, lengthy but elementary calculations give that

$$EL_{T,i}^2 \ge C_7, \quad \text{if } T \ge T^*, \tag{38}$$

where the constants C_7 and T^* only depend on $x_1, x_2, \ldots, x_K$ and $\lambda_1, \lambda_2, \ldots, \lambda_K$. On the other hand, applying again the Rosenthal inequality (cf. Petrov, 1995, p. 59) we obtain that

$$E|L_{T,i}|^{\kappa/2} \le C_8 T^{-\kappa/2}\{TE|\varepsilon_{i,0}|^\kappa + T^{\kappa/2}\}, \tag{39}$$

where C_8 only depends on the λ's. Using (38) and (39), we conclude that

$$\begin{aligned}\frac{\left(\sum_{i=1}^{N}E|L_{T,i}|^{\kappa/2}\right)^{2/\kappa}}{\left(\sum_{i=1}^{N}E(L_{T,i})^2\right)^{1/2}} &\le C_8\frac{\left(T^{1-\kappa/2}\sum_{i=1}^{N}E|\varepsilon_{i,0}|^\kappa\right)^{2/\kappa} + N^{2/\kappa}}{N^{1/2}}\\ &\le C_9 N^{(4-\kappa)/(2\kappa)}\left\{\left(T^{1-\kappa/2}\frac{1}{N}\sum_{i=1}^{N}E|\varepsilon_{i,0}|^\kappa\right)^{2/\kappa} + \left(\frac{1}{N}\sum_{i=1}^{N}E|\varepsilon_{i,0}|^\kappa\right)^{2/\kappa}\right\}\end{aligned}$$

on account of (11). Using Lyapunov's theorem (cf. DasGupta, 2008, p. 64), the asymptotic normality of $\sum_{1\le i\le N}L_{T,i}$ is established. Applying the Cramér-Wold rule (cf. DasGupta, 2008, p. 9), we obtain the convergence of the finite dimensional distributions to a normal law.

Next, we establish the covariance structure of the limit. Let $B(x)$ be a Brownian bridge. Following the arguments leading to (38), one can prove that for all x and y

$$\max_{1\le i\le N} |\mathrm{cov}(Q^2_{T,i}(x), Q^2_{T,i}(y)) - \mathrm{cov}(B^2(x), B^2(y))| \to 0.$$

Hence, $E\Gamma(x)\Gamma(y) = \mathrm{cov}(B^2(x), B^2(y))$. To obtain the representation of the limit in term of a Wiener process, we introduce the Ornstein–Uhlenback process $U(x)$, i.e. $U(x)$ is a stationary Gaussian process with $EU(x) = 0$ and $EU(x)U(y) = \exp(-|x-y|)$. It is well-known that

$$\left\{\frac{B(x)}{\sqrt{x(1-x)}}, 0<x<1\right\} \stackrel{\mathcal{D}}{=} \left\{U\left(\frac{1}{2}\log\left(\frac{x}{1-x}\right)\right), 0<x<1\right\}$$

(cf. Csörgő and Horváth, 1993, p. 255). By the stationarity of U, we have

$$\begin{aligned} E\left\{\frac{B^2(x)}{x(1-x)}\frac{B^2(y)}{y(1-y)}\right\} &= E\left\{U^2\left(\frac{1}{2}\log\left(\frac{x}{1-x}\right)\right)U^2\left(\frac{1}{2}\log\left(\frac{y}{1-y}\right)\right)\right\} \\ &= E\left\{U^2(0)U^2\left(\frac{1}{2}\log\left(\frac{y(1-x)}{x(1-y)}\right)\right)\right\} \end{aligned} \tag{40}$$

for all $0 < x \le y < 1$. It follows from the definition of U that for every $h > 0$, we have

$$(U(0), U(h)) \stackrel{\mathcal{D}}{=} (U(0), \exp(-h)U(0) + \sqrt{1-\exp(-2h)}\xi), \tag{41}$$

where ξ is a standard normal random variable, independent of $U(0)$. Using (41), elementary arguments yield that

$$E\{U^2(0)U^2(h)\} = 1 + 2\exp(-2h),$$

and therefore (40) implies that $E\Gamma(x)\Gamma(y) = x^2(1-y)^2$, if $0 \le x \le y \le 1$. Computing the covariance functions, it is easy to verify that Remark 1 holds.

We continue with the proof of the tightness. □

Lemma 2. *If* (4)–(9) *and* (12) *hold, then* $\bar V_{N,T}(x)$ *is tight in* $\mathcal{D}[0,1]$.

Proof. We use again (32). Applying Rosenthal's inequality (cf. Petrov, 1995, p. 59) we obtain for all $1 \le \ell \le k \le T$ that

$$\begin{aligned} A_1(k,\ell) &= E\left(\frac{1}{N^{1/2}}\sum_{i=1}^N \frac{1}{T\sigma_i^2}(\eta^2_{i,k} - \eta^2_{i,\ell} - E(\eta^2_{i,k} - \eta^2_{i,\ell}))\right)^{\kappa/2} \\ &\le C_1 N^{-\kappa/4}T^{-\kappa/2}\left\{\sum_{i=1}^N E(\eta^2_{i,k} - E\eta^2_{i,k})^{\kappa/2} + \left[\sum_{i=1}^N E(\eta^2_{i,k} - E\eta^2_{i,k})^2\right]^{\kappa/4}\right. \\ &\quad \left. + \sum_{i=1}^N E(\eta^2_{i,\ell} - E\eta^2_{i,\ell})^{\kappa/2} + \left[\sum_{i=1}^N E(\eta^2_{i,\ell} - E\eta^2_{i,\ell})^2\right]^{\kappa/4}\right\} \\ &\le C_2 N^{-\kappa/4}T^{-\kappa/2}\left\{\sum_{i=1}^N (E|\eta_{i,k}|^\kappa + E|\eta_{i,\ell}|^\kappa) + \left[\sum_{i=1}^N (E\eta^4_{i,k} + E\eta^4_{i,\ell})\right]^{\kappa/4}\right\}. \end{aligned}$$

First, we note that there is a constant C_3 depending only on $0 < \gamma \le \kappa$ such that

$$E|e^*_{i,t}|^\gamma < C_3 E|\varepsilon_{i,0}|^\gamma,$$

resulting in

$$E\eta^4_{i,k} \le C_4 E\varepsilon^4_{i,0} \quad \text{and} \quad E|\eta_{i,k}|^\kappa \le C_5 E|\varepsilon_{i,0}|^\kappa.$$

Hence,

$$A_1(k,\ell) \le C_6 T^{-\kappa/2}\left\{\frac{1}{N}\sum_{i=1}^N E|\varepsilon_{i,0}|^\kappa + \left[\frac{1}{N}\sum_{i=1}^N E\varepsilon^4_{i,0}\right]^{\kappa/4}\right\} \le C_7 T^{-\kappa/2}. \tag{42}$$

Repeating the arguments leading to (42) we obtain that

$$
\begin{aligned}
&A_2(k,\ell)\\
&= E\left\{\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left(Q_{T,i}\left(\frac{k}{T}\right)\eta_{i,k} - Q_{T,i}\left(\frac{\ell}{T}\right)\eta_{i,\ell} - E\left(Q_{T,i}\left(\frac{k}{T}\right)\eta_{i,k} - Q_{T,i}\left(\frac{\ell}{T}\right)\eta_{i,\ell}\right)\right)\right\}^{\kappa/2}\\
&\le C_8\frac{1}{(NT)^{\kappa/4}}\left\{\sum_{i=1}^{N}\left(E\left|Q_{T,i}\left(\frac{k}{T}\right)\eta_{i,k} - EQ_{T,i}\left(\frac{k}{T}\right)\eta_{i,k}\right|^{\kappa/2} + E\left|Q_{T,i}\left(\frac{\ell}{T}\right)\eta_{i,\ell} - EQ_{T,i}\left(\frac{\ell}{T}\right)\eta_{i,\ell}\right|^{\kappa/2}\right)\right.\\
&+\left[\sum_{i=1}^{N}E\left(Q_{T,i}\left(\frac{k}{T}\right)\eta_{i,k}\right) - EQ_{T,i}\left(\frac{k}{T}\right)\eta_{i,k}\right)^2\right]^{\kappa/4}\\
&+\left.\left[\sum_{i=1}^{N}E\left(Q_{T,i}\left(\frac{\ell}{T}\right)\eta_{i,\ell}\right) - EQ_{T,i}\left(\frac{\ell}{T}\right)\eta_{i,\ell}\right)^2\right]^{\kappa/4}\right\}\\
&\le C_9\frac{1}{(NT)^{\kappa/4}}\left\{\sum_{i=1}^{N}\left(E\left|Q_{T,i}\left(\frac{k}{T}\right)\eta_{i,k}\right|^{\kappa/2} + E\left|Q_{T,i}(\ell/T)\eta_{i,\ell}\right|^{\kappa/2}\right)\right.\\
&+\left.\left[\sum_{i=1}^{N}E\left(Q_{T,i}\left(\frac{k}{T}\right)\eta_{i,k}\right)^2\right]^{\kappa/4} + \left[\sum_{i=1}^{N}E\left(Q_{T,i}\left(\frac{\ell}{T}\right)\eta_{i,\ell}\right)^2\right]^{\kappa/4}\right\}.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, we get for all $0 < \gamma \le \kappa/2$, as in the proof of Lemma 1, that

$$
\begin{aligned}
E\left|Q_{T,i}\left(\frac{k}{T}\right)\eta_{i,k}\right|^{\gamma} &\le \left(E\left|Q_{T,i}\left(\frac{k}{T}\right)\right|^{2\gamma}E\left|\eta_{i,k}\right|^{2\gamma}\right)^{1/2}\\
&\le C_{10}\left\{\left(T^{-\gamma+1}E\left|\varepsilon_{i,0}\right|^{2\gamma}\right)^{1/2} + \left(E\varepsilon_{i,0}^2\right)^{\gamma/2}\right\}\left(E\left|\varepsilon_{i,0}\right|^{2\gamma}\right)^{1/2}\\
&\le C_{11}E|\varepsilon_{i,0}|^{2\gamma}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
A_2(k,\ell) &\le C_{12}\frac{1}{T^{\kappa/4}}\left\{\frac{1}{N}\sum_{i=1}^{N}E\left|\varepsilon_{i,0}\right|^{\kappa} + \left[\frac{1}{N}\sum_{i=1}^{N}E\varepsilon_{i,0}^4\right]^{\kappa/4}\right\}\\
&\le C_{13}\frac{1}{T^{\kappa/4}}.
\end{aligned}
$$

Next, we introduce

$$
A_3(k,\ell) = E\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{T,i}^2\left(\frac{\ell}{T}\right) - Q_{T,i}^2\left(\frac{k}{T}\right) - \left[\frac{\ell}{T}\left(1-\frac{\ell}{T}\right) - \frac{k}{T}\left(1-\frac{\ell}{T}\right)\right]\right)\right\}^{\kappa/2},
$$

$1 \le k \le \ell \le T$. It is easy to see that for all $0 < \gamma < \kappa/2$, we have

$$
\begin{aligned}
&E\left|\left(Q_{T,i}^2\left(\frac{\ell}{T}\right) - Q_{T,i}^2\left(\frac{k}{T}\right) - \left[\frac{\ell}{T}\left(1-\frac{\ell}{T}\right) - \frac{k}{T}\left(1-\frac{\ell}{T}\right)\right]\right)\right|^{\gamma}\\
&\le C_{15}E\left|Q_{T,i}^2\left(\frac{\ell}{T}\right) - Q_{T,i}^2\left(\frac{k}{T}\right)\right|^{\gamma} + C_{16}\left(\frac{\ell-k}{T}\right)^{\gamma}.
\end{aligned}
\tag{43}
$$

Using the Cauchy–Schwarz inequality, elementary arguments give

$$
\begin{aligned}
&E\left|Q_{T,i}^2\left(\frac{\ell}{T}\right) - Q_{T,i}^2\left(\frac{k}{T}\right)\right|^{\gamma}\\
&\le \left\{E\left(\left|Q_{T,i}\left(\frac{\ell}{T}\right)\right| + \left|Q_{T,i}\left(\frac{k}{T}\right)\right|\right)^{2\gamma}E\left|Q_{T,i}\left(\frac{\ell}{T}\right) - Q_{T,i}\left(\frac{k}{T}\right)\right|^{2\gamma}\right\}^{1/2}\\
&\le C_{17}\left\{E\left|Q_{T,i}\left(\frac{\ell}{T}\right)\right|^{2\gamma} + E\left|Q_{T,i}\left(\frac{k}{T}\right)\right|^{2\gamma}\right\}^{1/2}\left\{E\left|Q_{T,i}\left(\frac{\ell}{T}\right) - Q_{T,i}\left(\frac{k}{T}\right)\right|^{2\gamma}\right\}^{1/2}.
\end{aligned}
\tag{44}
$$

The Rosenthal inequality yields

$$
\begin{aligned}
E\left|Q_{T,i}\left(\frac{\ell}{T}\right)\right|^{2\gamma} &\le C_{18}T^{-\gamma}\left(T|\varepsilon_{i,0}|^{2\gamma} + \left(TE\varepsilon_{i,0}^2\right)^{\gamma}\right)\\
&\le C_{19}E|\varepsilon_{i,0}|^{2\gamma}
\end{aligned}
\tag{45}
$$

and similarly for any $\gamma \ge 1$

$$
\begin{aligned}
&E\left|Q_{T,i}\left(\frac{\ell}{T}\right)-Q_{T,i}\left(\frac{k}{T}\right)\right|^{2\gamma}\\
&\leq C_{20}T^{-\gamma}\left\{E\left|\sum_{j=k+1}^{\ell}\varepsilon_{i,j}\right|^{2\gamma}+\left(\frac{\ell-k}{T}\right)^{2\gamma}E\left|\sum_{j=1}^{T}\varepsilon_{i,j}\right|^{2\gamma}\right\}\\
&\leq C_{21}T^{-\gamma}\left\{(\ell-k)E\,|\,\varepsilon_{i,0}\,|^{2\gamma}+(\ell-k)^{\gamma}(E\varepsilon_{i,0}^2)^{\gamma}\right.\\
&\left.+\left(\frac{\ell-k}{T}\right)^{2\gamma}(TE|\varepsilon_{i,0}|^{2\gamma}+(TE\varepsilon_{i,0}^2)^{\gamma}\right\}\\
&\leq C_{22}\left(\frac{\ell-k}{T}\right)^{\gamma}E|\varepsilon_{i,0}|^{2\gamma}.
\end{aligned}
\tag{46}
$$

Since $A_3(k,\ell)$ is a sum of independent random variables, we can use again Rosenthal's inequality in conjection with the estimates in (43)–(46) with $\gamma=\kappa/2$ and $\gamma=2$ to conclude

$$
A_3(k,\ell)\leq c_{23}\left(\frac{\ell-k}{T}\right)^{\kappa/4}\frac{1}{N}\sum_{i=1}^{N}E|\varepsilon_{i,0}|^{\kappa}.
$$

The upper bounds for A_1, A_2 and A_3 imply that for all $1\leq k\leq\ell\leq T$

$$
E\,|\,\bar{V}_{N,T}\left(\frac{\ell}{T}\right)-\bar{V}_{N,T}\left(\frac{k}{T}\right)|^{\kappa/2}\leq C_{23}\left(\frac{\ell-k}{T}\right)^{\kappa/4}+C_{24}\frac{1}{T^{\kappa/4}}\leq C_{25}\left(\frac{\ell-k}{T}\right)^{\kappa/4},
$$

which yields

$$
E\,|\,\bar{V}_{N,T}(x)-\bar{V}_{N,T}(y)|^{\kappa/2}\leq C_{26}\,|\,x-y\,|^{\kappa/4}
$$

for all $0\leq x,y\leq 1$. Since $\kappa>4$, Lemma 2 follows from Theorem 12.3 of Billingsley (1968, p. 95). □

Proof of Theorem 1. *The result follows immediately from Lemmas* 1 *and* 2. □

6. PROOF OF THEOREM 3

Let

$$
v_{T,i}(x)=\begin{cases}-(T-t_0)\lfloor Tx\rfloor\delta_i/T, & \text{if } 0\leq\lfloor Tx\rfloor\leq t_0\\ -t_0(T-\lfloor Tx\rfloor)\delta_i/T, & \text{if } t_0<\lfloor Tx\rfloor\leq T.\end{cases}
$$

It is easy to see that

$$
\begin{aligned}
Z_{T,i}^2(x)=&\frac{1}{T}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right\}^2\\
&+\frac{2}{T}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right\}v_{T,i}(x)+\frac{1}{T}v_{T,i}^2(x).
\end{aligned}
$$

It follows from Theorem 1 that

$$
\sup_{0\leq x\leq 1}\left|\frac{1}{N^{1/2}}\sum_{i=1}^{N}\left[\frac{1}{\sigma_i^2}\frac{1}{T}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right\}^2-\frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right]\right|=O_P(1).
$$

Following the proofs in Section 5, one can show that

$$
\sup_{0\leq x\leq 1}\left|\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{v_{T,i}(x)}{T^{1/2}\sigma_i^2}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right\}\right|=O_P\left(\left(\frac{T}{N}\sum_{i=1}^{N}\delta_i^2\right)^{1/2}\right).
$$

Since

$$v_{T,i}^2\left(\frac{t_0}{T}\right) \geq cT^2\delta_i^2$$

with some positive c, Theorem 3 is proven.

7. PROOF OF THEOREM 4

To derive Theorem 4 from Theorem 1, we need to replace σ_i^2 with $\hat{\sigma}_T^2(i)$. The next provides some results for the kernel estimator of σ_i^2.

LEMMA 3. *If* (4)–(12) *and* (19)–(23) *hold, then we have*

$$|E\hat{\sigma}_T^2(i) - \tilde{\sigma}_T^2(i)| \leq C_1\frac{h}{T}E\varepsilon_{i,0}^2, \tag{47}$$

$$E(\hat{\sigma}_T^2(i) - \tilde{\sigma}_T^2(i))^2 \leq C_2\frac{h}{T}E\varepsilon_{i,0}^4, \tag{48}$$

$$E(\hat{\sigma}_T^2(i) - \tilde{\sigma}_T^2(i))^4 \leq C_3\left(\frac{h}{T}\right)^2 E\varepsilon_{i,0}^8, \tag{49}$$

$$|E[(\hat{\sigma}_T^2(i) - \tilde{\sigma}_T^2(i))Z_{T,i}^2(x)]| \leq C_4\frac{h}{T}E\varepsilon_{i,0}^4 \tag{50}$$

and

$$|\tilde{\sigma}_T^2(i) - \sigma_i^2| \leq C_5\frac{1}{h^\tau}E\varepsilon_{i,0}^4, \tag{51}$$

where $\tau = \min(\nu, \alpha - 1)$, $C_1, \ldots, C_5$ are constants and

$$\tilde{\sigma}_T^2(i) = Ee_{i,0}^2 + 2\sum_{s=1}^{T-1} K\left(\frac{s}{h}\right) Ee_{i,0}e_{i,s}.$$

PROOF. We can assume without loss of generality that $\mu_i = 0$. Elementary arguments give that with some constant c_1

$$\left|E\frac{1}{T}\sum_{t=1}^{T}(X_{i,t} - \bar{X}_T(i))^2 - E\frac{1}{T}\sum_{t=1}^{T}X_{i,t}^2\right| \leq c_1\frac{1}{T}E\varepsilon_{i,0}^2. \tag{52}$$

It is easy to see that

$$\hat{\gamma}_{T,s}(i) - \bar{\gamma}_{T,s}(i) = \frac{1}{T-s}\left\{-(T+s)\bar{X}_T^2(i) + \bar{X}_T(i)\sum_{t=1}^{s}X_{i,t} + \bar{X}_T(i)\sum_{j=T-s+1}^{T}X_{i,t}\right\}, \tag{53}$$

where

$$\bar{\gamma}_{T,s}(i) = \frac{1}{T-s}\sum_{t=1}^{T-s}X_{i,t}X_{i,t+s}.$$

Hence, similarly to (52), we have

$$|E\hat{\gamma}_{T,s}(i) - E\bar{\gamma}_{T,s}(i)| \leq c_2\frac{1}{T}E\varepsilon_{i,0}^2. \tag{54}$$

Now, (47) follows from (52)–(54).
Assumptions (4)–(6) and (8) yield that

$$E\left(\frac{1}{T}\sum_{t=1}^{T}(X_{i,t}-\bar{X}_T(i))^2-\frac{1}{T}\sum_{t=1}^{T}X_{i,t}^2\right)^2=E\bar{X}_T^4(i)\le c_3\frac{1}{T^2}E\varepsilon_{i,0}^4 \qquad (55)$$

with some constant c_3.

Using (59) we obtain

$$E\left\{\sum_{s=1}^{T-1}K\left(\frac{s}{h}\right)(\hat{\gamma}_{T,s}(i)-\bar{\gamma}_{T,s}(i))\right\}^2\le 8(A_{T,1}+A_{T,2}+A_{T,3}),$$

where

$$A_{T,1}=E\left\{\bar{X}_T^2(i)\sum_{s=1}^{T-1}K\left(\frac{s}{h}\right)\frac{T}{T-s}\right\}^2,$$

$$A_{T,2}=E\left\{\bar{X}_T(i)\sum_{s=1}^{T-1}K\left(\frac{s}{h}\right)\frac{1}{T-s}\sum_{t=1}^{s}X_{i,t}\right\}^2$$

and

$$A_{T,3}=E\left\{\bar{X}_T(i)\sum_{s=1}^{T-1}K\left(\frac{s}{h}\right)\frac{1}{T-s}\sum_{t=T-s+1}^{T}X_{i,t}\right\}^2.$$

It follows from (20) and (55) that

$$A_{T,1}\le c_4\left(\frac{h}{T}\right)^2E\varepsilon_{i,0}^4,$$

where c_4 is a constant. Similar arguments give that

$$A_{T,2}\le c_5\left(\frac{h}{T}\right)^2E\varepsilon_{i,0}^4\quad\text{and}\quad A_{T,3}\le c_6\left(\frac{h}{T}\right)^2E\varepsilon_{i,0}^4,$$

with some c_5 and c_6. Applying (4)–(6) and (8) one can easily verify that

$$E\left(\frac{1}{T-1}\sum_{t=1}^{T}[X_{i,t}^2-EX_{i,t}^2]\right\}^2\le\frac{c_7}{T}E\varepsilon_{i,0}^4.$$

Using again the linear structure of the $X_{i,t}$'s, we can write that

$$\begin{aligned}
&E\left\{\sum_{s=1}^{T-1}K\left(\frac{s}{h}\right)(\bar{\gamma}_{T,s}(i)-E\bar{\gamma}_{T,s}(i))\right\}^2\\
&=\sum_{s=1}^{T-1}\sum_{t=1}^{T-1}K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)E[\bar{\gamma}_{T,s}(i)\bar{\gamma}_{T,t}(i)-E\bar{\gamma}_{T,s}(i)E\bar{\gamma}_{T,t}(i)]\\
&=\sum_{s=1}^{T-1}\sum_{t=1}^{T-1}K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)\sum_{u=1}^{T-s}\sum_{v=1}^{T-t}\frac{1}{T-s}\frac{1}{T-t}A_i(u,v,s,t)\\
&=\sum_{s=1}^{T-1}\sum_{t=1}^{T-1}K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)\sum_{u=1}^{T-s}\sum_{v=1}^{T-t}\frac{1}{T-s}\frac{1}{T-t}\sum_{w=0}^{\infty}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}B_{u,v;i}(w,r,p,q),
\end{aligned}$$

where

$$A_i(u,v,s,t)=E[X_{i,u}X_{i,u+s}X_{i,v}X_{i,v+t}]-E[X_{i,u}X_{i,u+s}]E[X_{i,v}X_{i,v+t}]$$

and

$$B_{u,v;i}(w,r,p,q)=E[c_{i,w}\varepsilon_{u-w}c_{i,r}\varepsilon_{u+s-r}c_{i,p}\varepsilon_{v-p}c_{i,q}\varepsilon_{v+t-q}]-E[c_{i,w}\varepsilon_{u-w}a_{i,r}\varepsilon_{u+s-r}]E[(c_{i,p}\varepsilon_{v-p}c_{i,q}\varepsilon_{v+t-q})].$$

It is easy to see that $E(c_{i,w}\varepsilon_{u-w}c_{i,r}\varepsilon_{u+s-r})E(c_{i,p}\varepsilon_{v-p}c_{i,q}\varepsilon_{v+t-q})=0$ except if $r=w+s$ and $q=t+p$. Now, we consider the four cases when $B_{u,v;i}$ is different from 0. Case 1: Let $G_1(u,v)$ be the set of those w,r,p,q for which $u-w=u+s-r=v-p=v+t-q$. We note that on the set G_1 we have that $v=u+p-w, r=w+s, q=p+t$, so using (8) we obtain that

$$\sum_{u=1}^{T}\sum_{v=1}^{T}\sum_{G_1}|B_{u,v;i}(w,r,p,q)| \le c_8 T s^{-\alpha+1} t^{-\alpha+1} E\varepsilon_{i,0}^4.$$

Since $\alpha > 2$ and K is bounded, we obtain that

$$\left|\sum_{s=1}^{T-1}\sum_{t=1}^{T-1} K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)\sum_{u=1}^{T-s}\sum_{v=1}^{T-t}\frac{1}{T-s}\frac{1}{T-t}\sum_{G_1}B_{u,v;i}(w,r,p,q)\right| \le c_9\frac{1}{T}E\varepsilon_{i,0}^4.$$

Case 2: Let G_2 be the set on which $u-w=u+s-r \ne v-p=v+t-q$. By the independence of the $\varepsilon_{i,t}$'s, we have that $B_{u,v;i}=0$ on G_2 and therefore

$$\sum_{s=1}^{T-1}\sum_{t=1}^{T-1} K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)\sum_{u=1}^{T-s}\sum_{v=1}^{T-t}\frac{1}{T-s}\frac{1}{T-t}\sum_{G_2}B_{u,v;i}(w,r,p,q)=0.$$

Case 3: Let G_3 be the set on which $u-w=v-p \ne u+s-r=v+t-q$. On this set we have that $p=w+(v-u)$ and $q=r+(v-u)+(t-s)$. Hence, using again (8) we conclude

$$\begin{aligned}
&\sum_{s=1}^{T-1}\sum_{t=1}^{T-1} K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)\sum_{u=1}^{T-s}\sum_{v=1}^{T-t}\frac{1}{T-s}\frac{1}{T-t}\sum_{G_3}|B_{u,v;i}(w,r,p,q)|\\
&\le \frac{c_{10}}{T^2}\sum_{s=1}^{T-1}\sum_{t=1}^{T-1} K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)\sum_{u=1}^{T-s}\sum_{v=1}^{T-t}\sum_{w=1}^{\infty}\sum_{r=1}^{\infty} w^{-\alpha}r^{-\alpha}(1+|v-u|)^{-\alpha}(1+|t-s|)^{-\alpha}\\
&\le c_{11}\frac{h}{T}(E\varepsilon_{i,0}^2)^2.
\end{aligned}$$

Case 4: Let G_4 be the set on which $u-w=v+t-q \ne v-p=u+s-r$. On this set, we have that $p=w+(v-u)$ and $q=r+(v-u)+(t-s)$. Following the proof of Case 3, one can easily verify that

$$\sum_{s=1}^{T-1}\sum_{t=1}^{T-1} K\left(\frac{s}{h}\right)K\left(\frac{t}{h}\right)\sum_{u=1}^{T-s}\sum_{v=1}^{T-t}\frac{1}{T-s}\frac{1}{T-t}\sum_{G_4}|B_{u,v;i}(w,r,p,q)| \le c_{12}\frac{h}{T}(E\varepsilon_{i,0}^2)^2.$$

This also completes the proof of (48).

The proofs of (49) and (50) are very similar to that of (48), but there are more non-zero term, but the basic idea is the same and therefore the proofs of (49) and (50) are omitted.

We can assume that $a=1$ in (20). It follows from (20) and (8) that

$$\sum_{s=h}^{\infty}|E\varepsilon_{i,0}\varepsilon_{i,s}| \le c_{13}h^{-\alpha+1}E\varepsilon_{i,0}^2 \quad\text{and}\quad \sum_{s=ch}^{h}|E\varepsilon_{i,0}\varepsilon_{i,s}| \le c_{14}h^{-\alpha+1}E\varepsilon_{i,0}^2,$$

where $0 < c < 1$ is small enough but fixed. Since $\alpha > 1$, the Taylor expansion and (21) imply that

$$\sum_{s=1}^{ch}\left|K\left(\frac{s}{h}\right)-K(0)\right|E\varepsilon_{i,0}\varepsilon_{i,s} \le c_{15}\sum_{s=1}^{ch}\left(\frac{\ell}{h}\right)^{\nu}s^{-\alpha}E\varepsilon_{i,0}^2 \le c_{16}\frac{1}{h^{\nu}}E\varepsilon_{i,0}^2,$$

completing the proof of (51). □

Proof of Theorem 4. *According to Theorem 1, we need to show only*

$$\sup_{0\le x\le 1}|V_{N,T}(x)-\bar{V}_{N,T}(x)| = o_P(1). \tag{56}$$

Let $D_{N,T}$ denote the event that $\hat\sigma_T^2(i) \ge \sigma_i^2/2$ for all $1 \le i \le N$. By the independence of the panels and (49) we conclude that

$$\begin{aligned}
P\{D_{N,T}\} &= \prod_{i=1}^{N} P\{\hat\sigma_T^2(i) \ge \frac{\sigma_i^2}{2}\}\\
&\ge \prod_{i=1}^{N}(1-P\{|\hat\sigma_T^2(i)-\sigma_i^2| \ge \frac{\sigma_i^2}{2}\}\\
&\ge \prod_{i=1}^{N}\left\{1-\left(\frac{2}{\sigma_i^2}\right)^4 C_3\left(\frac{h}{T}\right)^2 E\varepsilon_{i,0}^8\right\}\\
&\to 1,
\end{aligned}$$

on account of assumptions (7) and (23). Hence, it is enough to establish (56) on $D_{N,T}$.

Next, we write

$$V_{N,T}(x) - \bar V_{N,T}(x) = \frac{1}{N^{1/2}}\sum_{i=1}^{N}\left(\frac{1}{\hat\sigma_T^2(i)} - \frac{1}{\sigma_i^2}\right)\left[Z_{T,i}^2(x) - \sigma_i^2 x(1-x)\right] + x(1-x)\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\hat\sigma_T^2(i)}.$$

Also,

$$\sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\hat\sigma_T^2(i)} = \sum_{i=1}^{N}\frac{E\hat\sigma_T^2(i) - \hat\sigma_T^2(i)}{\hat\sigma_T^2(i)} + \sum_{i=1}^{N}\frac{\sigma_i^2 - E\hat\sigma_T^2(i)}{\hat\sigma_T^2(i)}$$
$$= \sum_{i=1}^{N}\frac{E\hat\sigma_T^2(i) - \hat\sigma_T^2(i)}{\sigma_i^2} + \sum_{i=1}^{N}\frac{(E\hat\sigma_T^2(i) - \hat\sigma_T^2(i))^2}{\hat\sigma_T^2(i)\sigma_i^2} + \sum_{i=1}^{N}\frac{\sigma_i^2 - E\hat\sigma_T^2(i)}{\hat\sigma_T^2(i)}.$$

It follows form (47), (48) and the independence of the panels that

$$E\left(\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{E\hat\sigma_T^2(i) - \hat\sigma_T^2(i)}{\sigma_i^2}\right)^2 = \frac{1}{N}\sum_{i=1}^{N}\frac{(E\hat\sigma_T^2(i) - \hat\sigma_T^2(i))^2}{\sigma_i^4} \to 0$$

and similarly

$$\frac{1}{N^{1/2}}E\sum_{i=1}^{N}\frac{(E\hat\sigma_T^2(i) - \hat\sigma_T^2(i))^2}{\hat\sigma_T^2(i)\sigma_i^2}I\{D_{N,T}\} \le 2\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{E(E\hat\sigma_T^2(i) - \hat\sigma_T^2(i))^2}{\sigma_i^4}$$
$$\le c_1\frac{N^{1/2}h}{T}\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,0}^4 \to 0.$$

Using (47) and (48) we conclude

$$\frac{1}{N^{1/2}}\left|\sum_{i=1}^{N}E\frac{\sigma_i^2 - E\hat\sigma_T^2(i)}{\hat\sigma_T^2(i)}I\{D_{N,T}\}\right| \le 2\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{|\sigma_i^2 - E\hat\sigma_T^2(i)|}{\hat\sigma_T^2(i)}$$
$$\le c_2 N^{1/2}\left\{\frac{h}{T} + \frac{1}{h^\tau}\right\}\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,0}^4 \to 0,$$

completing the proof of

$$\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\hat\sigma_T^2(i)}x(1-x) = o_P(1). \tag{57}$$

Following the proof of Lemma 2, one can verify that

$$\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\sigma_i^2\hat\sigma_T^2(i)}[Z_{T,i}^2(x) - \sigma_i^2 x(1-x)] \text{ is tight in } \mathcal{D}[0,1],$$

so we need to prove only that

$$\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\sigma_i^2\hat\sigma_T^2(i)}Z_{T,i}^2(x) - \frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\hat\sigma_T^2(i)}x(1-x) = o_P(1) \tag{58}$$

for all $0 \le x \le 1$. On account of (57), we need to consider the first term in (58). We use the decomposition

$$\sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\sigma_i^2\hat\sigma_T^2(i)}Z_{T,i}^2(x) = \sum_{i=1}^{N}\frac{\sigma_i^2 - \hat\sigma_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x) + \sum_{i=1}^{N}\frac{(\sigma_i^2 - \hat\sigma_T^2(i))^2}{\sigma_i^4\hat\sigma_T^2(i)}Z_{T,i}^2(x)$$
$$= \sum_{i=1}^{N}\frac{\tilde\sigma_T^2(i) - \hat\sigma_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x) + \sum_{i=1}^{N}\frac{\sigma_i^2 - \tilde\sigma_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)$$
$$+ \sum_{i=1}^{N}\frac{(\sigma_i^2 - \hat\sigma_T^2(i))^2}{\sigma_i^4\hat\sigma_T^2(i)}Z_{T,i}^2(x).$$

It follows from (50) and (23) that

$$\frac{1}{N^{1/2}}\sum_{i=1}^{N}\left|E\frac{\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)\right|\le c_3N^{1/2}\frac{h}{T}\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,0}^4\to 0.$$

By the independence of the panels, we have that

$$E\left(\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)-E\frac{\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)\right]\right)^2$$
$$=\frac{1}{N}\sum_{i=1}^{N}\operatorname{var}\left(\frac{\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)\right).$$

Assumptions (6) and (8) yield that $EZ_{T,i}^8\le c_4E\varepsilon_{i,0}^8$, the Cauchy–Schwarz inequality with (49) implies that

$$\begin{aligned}\operatorname{var}\left(\frac{\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)\right)&\le\frac{1}{\sigma_i^8}E\{(\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i))^2Z_{T,i}^4(x)\}\\&\le\frac{1}{\sigma_i^8}(E(\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i))^4)^{1/2}(EZ_{T,i}^8)^{1/2}\\&\le c_5\frac{h}{T}E\varepsilon_{i,0}^8.\end{aligned}$$

Thus, we have

$$\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\tilde{\sigma}_T^2(i)-\hat{\sigma}_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)=o_P(1)$$

for all $0\le x\le 1$. Similarly,

$$\begin{aligned}E\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{(\sigma_i^2-\hat{\sigma}_T^2(i))^2}{\sigma_i^4\hat{\sigma}_T^2(i)}Z_{T,i}^2(x)I\{D_{N,T}\}&\le c_6\frac{1}{N^{1/2}}\sum_{i=1}^{N}(E(\sigma_i^2-\hat{\sigma}_T^2(i))^4)^{1/2}(EZ_{T,i}^4(x))^{1/2}\\&\le c_7N^{1/2}\frac{h}{T}\frac{1}{N}\sum_{i=1}^{N}E\varepsilon_{i,0}^8\to 0.\end{aligned}$$

We already showed that $EZ_{T,i}^2(x)\le c_8E\varepsilon_{i,0}$, so (51) and (7) yield

$$\begin{aligned}\frac{1}{N^{1/2}}E\left|\sum_{i=1}^{N}\frac{\sigma_i^2-\tilde{\sigma}_T^2(i)}{\sigma_i^4}Z_{T,i}^2(x)\right|&\le\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{|\sigma_i^2-\tilde{\sigma}_T^2(i)|}{\sigma_i^4}EZ_{T,i}^2(x)\\&\le c_9\frac{N^{1/2}}{h^{\tau}},\end{aligned}$$

completing the proof of (58). □

8. PROOFS OF (28)–(30)

Using (25) we can write

$$\begin{aligned}\bar{V}_{N,T}(x)=&\frac{1}{N^{1/2}}\sum_{i=1}^{N}\left\{\frac{1}{\sigma_i^2}\frac{1}{T}\left[\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right]^2-\frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right\}\\&+\frac{2}{N^{1/2}}\left\{\sum_{i=1}^{N}\frac{\phi_i}{\sigma_i^2}\frac{1}{T^{1/2}}\left[\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right]\right\}\frac{1}{T^{1/2}}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}\zeta_t-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}\zeta_t\right\}\\&+\frac{1}{T}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}\zeta_t-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}\zeta_t\right\}^2\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\phi_i^2}{\sigma_i^2}.\end{aligned}$$

It follows from (27) that

$$\frac{1}{T^{1/2}}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}\zeta_t-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}\zeta_t\right\}\xrightarrow{\mathcal{D}[0,1]}B(x),\qquad(59)$$

where $B(x)$ is a Brownian bridge. Following the proofs in Section 5, one can show that

$$\sup_{0\le x\le 1}\left|\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\phi_i}{\sigma_i^2}\frac{1}{T^{1/2}}\left[\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right]\right|=O_P(1) \tag{60}$$

and therefore Theorem 1 implies (28). If $\max_{1\le i\le N}\phi_{i,N}\to 0$, then in this case (60) can be replaced with

$$\sup_{0\le x\le 1}\left|\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\phi_i}{\sigma_i^2}\frac{1}{T^{1/2}}\left[\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right]\right|=o_P(1). \tag{61}$$

Hence, (29) follows from Theorem 1 and (59). To prove (30), we note that

$$\begin{aligned}&\frac{1}{N^{1/2}}\sum_{i=1}^{N}\left\{\frac{1}{\sigma_i^2+\phi_i^2}Z_{T,i}^2(x)-\frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right\}\\&=\frac{1}{N^{1/2}}\sum_{i=1}^{N}\left\{\frac{1}{\sigma_i^2}\frac{1}{T}\left[\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right]^2-\frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right\}\frac{\sigma_i^2}{\sigma_i^2+\phi_i^2}\\&+\frac{2}{N^{1/2}}\left\{\sum_{i=1}^{N}\frac{1}{\sigma_i^2+\phi_i^2}\frac{1}{T^{1/2}}\left[\sum_{t=1}^{\lfloor Tx\rfloor}e_{i,t}-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}e_{i,t}\right]\right\}\frac{1}{T^{1/2}}\left\{\sum_{t=1}^{\lfloor Tx\rfloor}\zeta_t-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}\zeta_t\right\}\\&+\left\{\frac{1}{T}\left[\sum_{t=1}^{\lfloor Tx\rfloor}\zeta_t-\frac{\lfloor Tx\rfloor}{T}\sum_{t=1}^{T}\zeta_t\right]^2-\frac{\lfloor Tx\rfloor(T-\lfloor Tx\rfloor)}{T^2}\right\}\frac{1}{N^{1/2}}\sum_{i=1}^{N}\frac{\phi_i^2}{\sigma_i^2+\phi_i^2}.\end{aligned}$$

Repeating the arguments leading to (29), one can establish (30).

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